J. H. KIM
Research Fellow,
Department of Mechanical
Engineering.

A. J. ACOSTA
Professor of Mechanical
Engineering. Mem. ASME

California Institute of
Technology, Pasadena, Calif.

# A Note on the Unsteady Cavity Flow in a Tunnel

*The unsteady internal cavitating flow such as the one observed in a pump or a turbine is studied for a simple two-dimensional model of a base-cavitating wedge in an infinite tunnel and it is shown how the cavitation compliance can be calculated using the linearized free streamline theory. Numerical values are obtained for the limiting case of a free jet. Two important features are: First, the cavitation compliance is found to be of complex form, having additional resistive and reactive terms beyond the purely inertial oscillation of the whole channel in "slug flow." Second, the compliance has a strong dependence on frequency.*

## Introduction

Liquid filled hydraulic systems often operate in such a way that cavitation may take place in one or more of the components of the system. Most often the cavitation will occur in a pump or a turbine as the liquid velocity there is usually the greatest in these devices. Because of the basic unsteady characteristics of the cavitation process itself, plus the presence of the piping walls, an unsteady internal cavity flow such as occurring in these devices is more difficult to analyze than either a steady flow in a channel or an unsteady flow in an infinite medium. For example, because of the dynamics of the liquid motion in the piping of the pump or turbine, it may be possible for a system instability to occur. A good example of this kind of instability is the thrust instability sometimes observed in liquid rocket engines which is due to a coupling between the feed pump, the thrust of the engine, the supporting structure and the motion of the propellant in the feed lines to the pump. This kind of instability is described by the name "POGO" instability and has been a subject of considerable interest [1, 2, 3].[1]

Typically in these works, the two types of assumptions made are the following. First, a passive "compliance" is attributed to the presence of cavitation in the pump, that is, the presence of cavitation is visualized as acting like a pressurized reservoir, and numerical values for the pump cavitation compliance are determined from dynamic experiments, or test stand firings, to make the observed frequencies match the theoretical ones.

Second, the behavior of the pump during the unsteady motions of the transient flows is assumed to be quasi-steady, namely, the change in pump performance parameters with flow rate and inlet pressure are assumed to be those which occur for steady state operation. This type of analysis is certainly the correct one for a first step, especially if the frequencies of oscillation are sufficiently low. On the other hand, it should be pointed out that the aforementioned inherent unsteadiness of cavitating flows is not taken into account.

In this paper, we present how the dynamic analysis of unsteady internal cavitating flows can be made using the linearized free streamline theory and thus how the theoretical values for cavitation compliance could be obtained.

As a simple example, it is instructive to consider the following model situation in a water tunnel as a representative problem.

## Formulation and Solution of the Problem

Consider a base-cavitating symmetrical wedge of unit length and a small half apex angle $\gamma$ placed in the middle of a two-dimensional tunnel of height $2h$. The flow is assumed to be two-dimensional, incompressible, inviscid and irrotational. The velocities up and downstream infinity are specified as $U + \tilde{u}_1 e^{i\omega t}$ and $U + \tilde{u}_2 e^{i\omega t}$, respectively, $U$ being the steady velocity and $\frac{|\tilde{u}_1|}{U}, \frac{|\tilde{u}_2|}{U} << 1$. The cavity pressure is assumed to be a given function of time only. The cavity length will be assumed to be time independent, although we will allow the volume of the cavity to vary with time. This assumption was employed by Leehey [4] in treating a problem of unsteady cavity flow in an infinite fluid. When we are only interested in the volume fluctuation of the cavity and not in the actual length of the cavity, as in the present problem, this assumption is certainly a reasonable one. In this way, the unsteady flow and the steady flow become separable. The flow configuration is illustrated in Fig. 1.

Now, let us write the velocity vector as $\mathbf{q} = (U + u, v)$, where $\frac{|u|}{U}, \frac{|v|}{U} << 1$. Then, retaining only the terms of first order in

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[1]**Numbers in brackets designate References at end of paper.**

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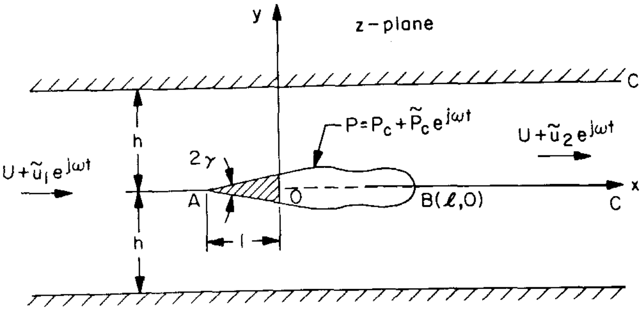


**Fig. 1 A simple model for unsteady internal cavitating flow**

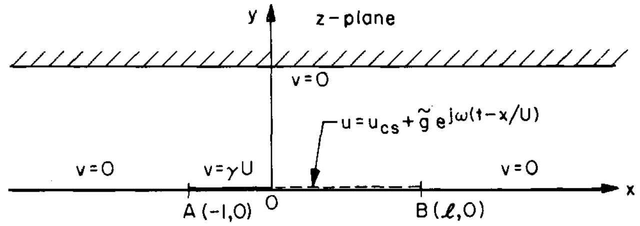


**Fig. 2 Linearized boundary conditions for velocity components in *z*-plane**

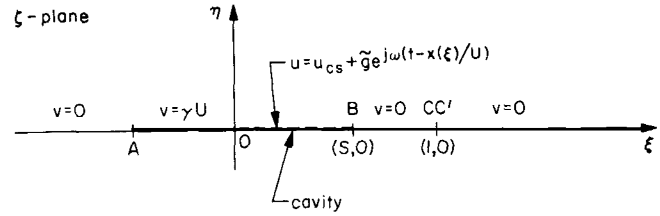


**Fig. 3 Boundary conditions in the upper half of the transformed plane**

$u$, $v$, and $\gamma$, we obtain the following linearized boundary conditions:

$v = 0$ on the wall and on the real axis outside the cavity-body.
$v = \gamma U$ on the upper wedge surface.
$v = -\gamma U$ on the lower wedge surface.
$u = u_{cs} + \tilde{g}e^{i\omega(t-x/U)}$ on the cavity.

The boundary condition on the cavity was obtained from $\dfrac{\partial u}{\partial t} + U\dfrac{\partial u}{\partial x} = 0$ on the cavity, from which writing $u = u_{cs} + \tilde{u}_c e^{j\omega t}$ where $u_{cs}$ is the steady velocity on the cavity, one obtains $u = u_{cs} + \tilde{g}e^{j\omega(t-x/U)}$ where $\tilde{g}$ is an unknown constant. Assuming the cavity-body to be thin, we can write the boundary conditions on the linearized $z$-plane as in Fig. 2. Because of the symmetry of the flow, only the upper-half portion of the flow field is considered. The upper half of the flow can be mapped into the upper half of another plane ($\zeta$-plane) by the transformation

$$z = -\frac{h}{\pi}\log(\zeta - 1) + ih \tag{1}$$

The end of the cavity maps into $(S, 0)$ in the $\zeta$-plane where $l = -\dfrac{h}{\pi}\log(1 - S)$. Fig. 3 shows the appropriate boundary conditions in the $\zeta$-plane. If we write $w = u - iv = w_s(\zeta) + \tilde{w}(\zeta)e^{j\omega t}$ where $w_s(\zeta)$ is the steady complex velocity, we obtain the following boundary values for $\tilde{u}$ and $\tilde{v}(\tilde{w} = \tilde{u} - i\tilde{v})$ along the real axis of the $\zeta$-plane:

$$\tilde{v} = 0, \qquad \xi < 0$$

$$\tilde{u} = \tilde{g}e^{-j\frac{\omega}{U}x(\xi)}, \quad 0 < \xi < S$$

$$\tilde{v} = 0, \qquad \xi > S.$$

This is a mixed-type Hilbert boundary value problem and the solution is obtained following the methods described in references [5] or [6]. The solution is found to be

$$\tilde{w}(\zeta) = \sqrt{\frac{\zeta}{\zeta - S}}\left\{\frac{\tilde{g}}{\pi}\int_0^S \sqrt{\frac{S-\xi}{\xi}}\,\frac{(1-\xi)^{i\beta}}{\xi - \zeta}\,d\xi + \tilde{u}_1\right\} \tag{2}$$

where

$$\beta = \frac{\omega h}{\pi U}$$

As $\zeta \to 1$, we must have $\tilde{w} \to \tilde{u}_2$. This condition gives

$$\tilde{g} = \frac{\pi(\tilde{u}_1 - \sqrt{1-S}\,\tilde{u}_2)}{\displaystyle\int_0^S \sqrt{\frac{S-\xi}{\xi}}\,(1-\xi)^{i\beta-1}d\xi}. \tag{3}$$

The relation between the pressure and velocity up and downstream infinity may be obtained from this solution which will in turn ultimately enable us to calculate the cavitation compliance.

From the linearized Euler's equation, it is easy to find that

$$p \to -j\omega\rho\tilde{u}_1 x e^{j\omega t} + \tilde{p}_1 e^{j\omega t} + p_{s\infty}$$
$$\text{as } x \to -\infty.$$

Here, $\tilde{p}_1$ may be thought of as the residual pressure fluctuation in addition to that of the oscillation of the channel fluid as a "slug." $p_{s\infty}$ is the steady pressure at infinity. Similarly, for the downstream pressure,

$$p \to -j\omega\rho\tilde{u}_2 x e^{j\omega t} + \tilde{p}_2 e^{j\omega t} + p_{s\infty}$$
$$\text{as } x \to +\infty.$$

It is desirable to find the relation between $(\tilde{p}_1, \tilde{p}_2)$ and $(\tilde{u}_1, \tilde{u}_2)$ as this is the essential additive dynamic effect to the system. For this purpose, let us integrate the linearized Euler's equation along the upper wall, obtaining

## Nomenclature

$\beta$ = dimensionless frequency, $\beta = \dfrac{\omega h}{\pi U}$
$\tilde{g}$ = unsteady part of the $x$-velocity on the cavity
$h$ = half height of the tunnel
$i$ = unit imaginary number with respect to space, $ij \neq -1$
$j$ = unit imaginary number with respect to time, $ij \neq -1$
$l$ = length of the cavity
$p$ = pressure
$p_1, p_2$ = residual oscillatory pressure at up and downstream infinity, respectively
$R$ = $-\dfrac{\tilde{p}_c - \tilde{p}_1}{\rho U\tilde{u}_1}$, nondimensionalized residual pressure difference between free surface and upstream infinity for a free jet
$\rho$ = density of the fluid
$t$ = time
$U$ = steady velocity at infinity
$\tilde{u}_1, \tilde{u}_2$ = unsteady part of the velocity at up and downstream infinity, respectively
$u, v$ = small perturbation velocities in the $x$ and $y$-direction, respectively
$\tilde{u}, \tilde{v}$ = unsteady parts of $u$ and $v$, respectively
$w$ = $u$-$iv$, complex velocity
$\omega$ = angular velocity

$$U(\tilde{u}_2 - \tilde{u}_1) - j\omega \int_{-\infty}^{\infty} x \frac{d\tilde{u}(x, h)}{dx}\, dx = -\frac{1}{\rho}(\tilde{p}_2 - \tilde{p}_1). \qquad (4)$$

Similarly, integrating the Euler's equation along the real axis results in

$$(U\tilde{g}\sqrt{1+\sigma} - \tilde{u}_1) - j\omega \int_{-\infty}^{0} x \frac{d\tilde{u}(x, 0)}{dx}\, dx = -\frac{1}{\rho}(\tilde{p}_c - \tilde{p}_1) \qquad (5)$$

where $\sigma$ is the cavitation number defined by

$$\sigma = \frac{p_{s\infty} - p_c}{\frac{1}{2}\rho U^2}.$$

Equations (4) and (5) give the desired relation between $(\tilde{p}_1, \tilde{p}_2)$ and $(\tilde{u}_1, \tilde{u}_2)$.

**Limiting Case.** **A simple situation is obtained when the flow becomes choked. In this case, the cavity extends to downstream infinity and thus it corresponds to a free jet emanating from a channel. Practically, we must have $\tilde{p}_c = 0$ for a free jet, but we will retain this term throughout for the purpose of generality.**

The unsteady velocity on the free surface is $\tilde{g}e^{-j\frac{\omega}{U}x}$ and will propagate downstream so that $\tilde{u}_2$ is indeterminate in this example. That is, we will have $\tilde{u}_2 \propto \tilde{g}e^{-j\frac{\omega}{U}x}$ as $x \to +\infty$.

For this choked limit, $S \to 1$ and the solution becomes

$$\tilde{w}(\zeta) = -\frac{\tilde{u}_1}{B}\sqrt{\zeta(\zeta-1)} \int_0^1 \frac{\xi^{-1/2}(1-\xi)^{j\beta-1/2}}{\xi - \zeta}\, d\xi$$

where

$$B = B\left(\frac{1}{2} + j\beta, \frac{1}{2}\right) = \int_0^1 \xi^{-1/2}(1-\xi)^{j\beta - 1/2} d\xi$$

is a beta function. We also have $\tilde{g} = \dfrac{\pi \tilde{u}_1}{B}$.

For the simplest case of $\gamma \to 0$, $\sigma$ will also approach zero and equation (5) now becomes

$$-\frac{\tilde{p}_c - \tilde{p}_1}{\rho U \tilde{u}_1} = \frac{j\beta}{B}\int_{-\infty}^{0} \log(1-\xi)\frac{d}{d\xi}$$
$$\left\{\sqrt{\xi(\xi-1)}\int_0^1 \frac{\tau^{-1/2}(1-\tau)^{j\beta-1/2}}{\tau - \xi}\, d\tau\right\} d\xi + \frac{\pi}{B} - 1. \qquad (6)$$

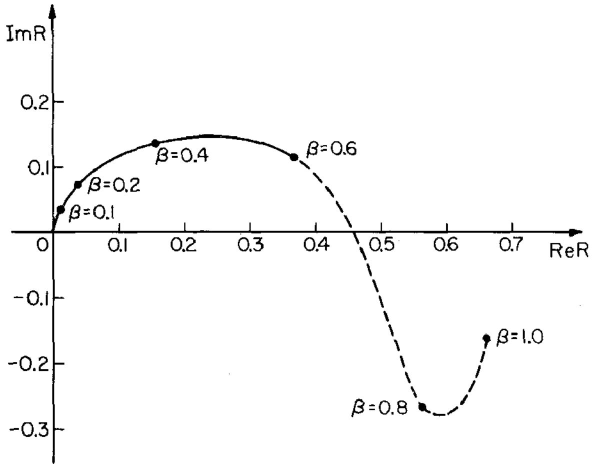


**Fig. 4 Vector diagram for $R$ as a function of $\beta$**

For this free jet case, some numerical values of $R \equiv -\dfrac{\tilde{p}_c - \tilde{p}_1}{\rho U \tilde{u}_1}$ are calculated from equation (6) and the vector diagram is given as a function of $\beta$ in Fig. 4. The corresponding numerical values are tabulated in the Appendix.

As a demonstration of how to apply $R$ to a practical problem, let us consider the example illustrated in Fig. 5. Here, we have an unsteady free jet coming out of a long channel of length $L$ and height $2h$. The upstream end of the channel is connected to a very large reservoir and the pressure on the free surface of the reservoir is given together with the pressure on the free jet. Our task is to find the fluctuating velocity $\tilde{u}_1$ in the channel as a function of frequency.

Since $L >> 2h$, we may approximate $p_A(t) \cong p_{As} + jL\rho\omega\tilde{u}_1 e^{j\omega t} + \tilde{p}_A e^{j\omega t}$, where the subscript $A$ denotes the junction between the reservoir and the tunnel and $p_{As}$ is the steady pressure there.

The Bernoulli's equation between the reservoir and the point $A$ can be written

$$\frac{p_0 + \tilde{p}_0 e^{j\omega t}}{\rho} + g y_0 = \frac{p_{As} + L\rho j\omega \tilde{u}_1 e^{j\omega t} + \tilde{p}_A e^{j\omega t}}{\rho} + \frac{1}{2}(U + \tilde{u}_1 e^{j\omega t})^2 + \left.\frac{\partial \varphi}{\partial t}\right|_A$$

where $y_0$ is the elevation of the reservoir surface relative to the channel and $\varphi$ is the velocity potential. Extracting the unsteady part, retaining the linear terms only and introducing the "effective length" $L_0$ such that

$$L_0 j\omega \tilde{u}_1 e^{j\omega t} = Lj\omega\tilde{u}_1 e^{j\omega t} + \left.\frac{\partial \varphi}{\partial t}\right|_A,$$

we may write

$$\frac{\tilde{p}_0}{\rho} = L_0 j\omega \tilde{u}_1 + \frac{\tilde{p}_A}{\rho} + U\tilde{u}_1$$

or, using

$$R \cong \frac{\tilde{p}_c - \tilde{p}_A}{\rho U \tilde{u}_1}$$

and solving for $\tilde{u}_1$, we obtain

$$\tilde{u}_1 = \frac{\tilde{p}_0 - \tilde{p}_c}{\rho U\left(1 + R + j\beta\dfrac{\pi L_0}{h}\right)}. \qquad (7)$$

For $R = 0$, namely $\tilde{p}_c = \tilde{p}_A$, one obtains

$$\tilde{u}_1 = \frac{\tilde{p}_0 - \tilde{p}_c}{\rho U\left(1 + j\beta\dfrac{\pi L_0}{h}\right)}$$

which will just be the case when the channel flow and the free

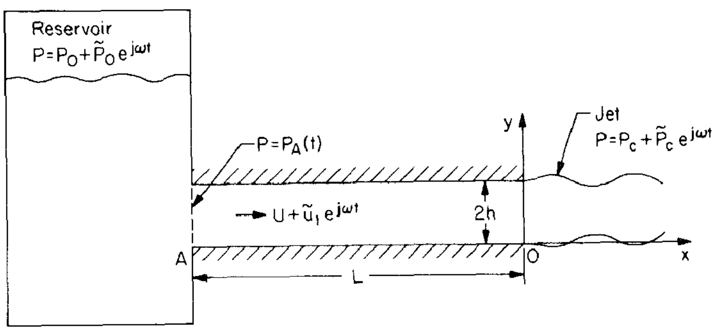


**Fig. 5 A jet flow emanating from a long channel connected to a large reservoir**

jet are thought of as being a "slug." Therefore, $R$ is clearly seen to be a measure of the residual dynamic effect added to the oscillation of the channel flow as a slug. This residual effect is small when the frequency of oscillation is low. For a numerical example of the case of $\tilde{p}_c = 0$, $\tilde{p}_0 = 2$ percent of the atmospheric pressure, $L_0 = 10$ ft, $\rho = 62.4$ lb/ft$^3$, $U = 30$ ft/sec, $h = 0.5$ ft, it can be shown that the difference in values of $\tilde{u}_1$ with and without the residual effect is less than 2 percent for values of $\beta$ up to 0.1 which roughly corresponds to a frequency $\omega$ of 20 cycles/sec.

## Concluding Remarks

Since the cavity compliance is proportional to $\dfrac{\tilde{u}_1 - \tilde{u}_2}{\tilde{p}_1 - \tilde{p}_c}$ [2, 7] it is obvious that the cavity compliance must have a complex form because $R \propto \dfrac{\tilde{p}_c - \tilde{p}_1}{\tilde{u}_1}$ is found to be complex. For the special case of the choked cavity treated above, a type of compliance similar to that of reference [7] can be defined in terms of $R$ as

$$K^* = \frac{1}{2\pi j\beta R}$$

except that the fluctuating downstream term $\tilde{u}_2$ is deleted.

The usefulness of $R$ or $K^*$ above is that it permits one to make dynamic analyses of unsteady flow in channels terminating in a free jet. Its importance in the present problem is that it is *complex*: that is, the additional dynamic response beyond the purely inertial oscillation of the whole channel in "slug flow" has resistive and reactive terms. Another importance is that it shows a strong dependence on frequency.

This leads us to believe that the corresponding cascade problem which models the actual cavitating inducer will have a similar frequency dependent compliance which we anticipate to be complex. The cascade problem is essentially not different from the channel flow problem and is also currently under study by the authors using a similar analysis to the one presented here.


## Acknowledgments

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# APPENDIX

## Numerical Values of $R(R = |R|\,e^{i\theta})$ for the Free Jet

| $\beta$ | $ReR$ | $ImR$ | $\|R\|$ | $\theta$ (degree) |
|---|---|---|---|---|
| 0.001 | $-0.9642\times10^{-3}$ | $0.3751\times10^{-3}$ | $0.1035\times10^{-2}$ | 158.74 |
| 0.002 | $-0.9616\times10^{-3}$ | $0.7503\times10^{-3}$ | $0.1220\times10^{-2}$ | 142.04 |
| 0.004 | $-0.9510\times10^{-3}$ | $0.1501\times10^{-2}$ | $0.1777\times10^{-2}$ | 122.37 |
| 0.006 | $-0.9334\times10^{-3}$ | $0.2251\times10^{-2}$ | $0.2437\times10^{-2}$ | 112.52 |
| 0.008 | $-0.9087\times10^{-3}$ | $0.3001\times10^{-2}$ | $0.3136\times10^{-2}$ | 106.85 |
| 0.01 | $-0.8770\times10^{-3}$ | $0.3751\times10^{-2}$ | $0.3852\times10^{-2}$ | 103.16 |
| 0.02 | $-0.6124\times10^{-3}$ | $0.7502\times10^{-2}$ | $0.7527\times10^{-2}$ | 94.67 |
| 0.04 | $0.4472\times10^{-3}$ | $0.1500\times10^{-1}$ | $0.1501\times10^{-1}$ | 88.29 |
| 0.06 | $0.2219\times10^{-2}$ | $0.2250\times10^{-1}$ | $0.2261\times10^{-1}$ | 84.37 |
| 0.1 | $0.7930\times10^{-2}$ | $0.3745\times10^{-1}$ | $0.3828\times10^{-1}$ | 78.04 |
| 0.2 | $0.3557\times10^{-1}$ | $0.7424\times10^{-1}$ | $0.8232\times10^{-1}$ | 64.4 |
| 0.4 | 0.1557 | 0.1336 | 0.2067 | 40.26 |
| 0.6 | 0.3682 | 0.1162 | 0.3861 | 17.51 |
| 0.8 | 0.5606 | −0.2638 | 0.5612 | −2.69 |
| 1.0 | 0.6581 | −0.1604 | 0.6774 | −13.70 |

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